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On a Physical Interpretation of the Schwarzschild Metric

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Abstract

A method is devised for giving a physical interpretation to the customary Schwarzschild coordinates in the vicinity of a charged or uncharged isolated mass. The construction is accomplished by introducing systems that are allowed to freely fall in toward the mass from infinity (drift-systems). It is demonstrated that the Schwarzschild spatial coordinates and their increments have a full physical significance in terms of rod and clock measurements performed in the drift-systems. The time coordinate and its increment are not so amenable to treatment and cannot be considered as having been given such physical significance. In the discussion the Schwarzschild metric about an uncharged and charged mass is derived, in part, by heuristic classical arguments employing conservation of energy. The arguments are then shown to be valid by consulting the Field Equations. In the derivation the gravitational singularity (at $2GM/C^2$) takes on the significance of being the location at which a drift-system achieves the speed of light relative to a proper system at the same point.

1. *Introduction*

One of the outstanding contributions to general relativity is Schwarzschild's solution for the metric about an isolated mass. This is so, of course, since the three 'crucial tests' of the General Theory consist of experiments executed in situations described by Schwarzschild's metric. In connection with this confirmation it has been pointed out by several authors (Eriksson, 1960; Balazs, 1959; Schiff, 1960; Schild, 1960) that the Schwarzschild solution can also be obtained without the full apparatus of the General Theory—that is, without the use of the Field Equations. These investigations are important for two reasons: (1) they perhaps demonstrate the uncomfortable fact that the successes of general relativity can also be explained by other (less comprehensive) theories; (2) these other constructions apparently lead, by their very nature, to the physical significance of the Schwarzschild coordinate system.

It is with the latter aspect that the present paper is concerned. Specifically, we are concerned with an explication of the physical significance of the Schwarzschild metric *within the framework of the Field Equations*. Despite the fact that problem area (2) has been discussed in the literature as mentioned, it is felt that these discussions are not sufficient for our purpose since, firstly, their purpose is to construct alternate (to the Field Equations) procedures for constructing the Schwarzschild metric. Consequently, they do not demonstrate whether or not their postulates are consequences of the Field Equations; secondly, these discussions are usually somewhat sketchy and do not, in our opinion, take due account of some of the subtle points involved. These points will be mentioned at their appropriate places in the context of the paper. We only note one example now; the heretofore unmentioned fact that the treatment of the Schwarzschild time coordinate in the references cited is incomplete.

Actually, the problem under consideration here is a special case of the general problem of devising physical interpretations for the coordinates used in general relativity. In general, when a problem is solved via the Field Equations one obtains the metric $g_{\mu\nu}$ expressed as a function of coordinates $(x^1 \dots x^4)$ whose physical significance is unknown. More explicitly, one can *relate* the coordinate differentials (dx^μ) to the results of rod and clock measurements occurring in a local inertial system at the point of interest, but this in no way constitutes an *interpretation* of these differentials in terms of rod and clock measurements relative to some particular observer. That is, in order to give physical significance to the (dx^μ) one must find an observer such that when two neighboring events occur, the observer can, by means of rod and clock measurements, directly determine (without reference to another observer's measurements) the associated differentials.

In the following discussion a method will be given for constructing the Schwarzschild line-element which at the same time gives it a physical interpretation as defined above. The method used demands the appearance of a gravitational singularity whose physical significance thus emerges naturally in the course of the derivation.

The following considerations are divided into essentially two parts. The first is concerned with the Schwarzschild line-element about a symmetric uncharged mass, and the second has to do with similar considerations concerning a symmetric charged mass. In each part the following program is followed: using the principle of strong equivalence one proceeds as far as possible toward the construction of the line-element using rod and clock measurements in space-time. A point is reached where an heuristic argument is invoked to yield the desired results. The Field Equations are then consulted and it is shown that the heuristic arguments are indeed correct. Thereby the entire consideration is brought within the framework of the General Theory as it is usually conceived. Although the heuristic parts of the considerations could have been omitted by appealing directly to the Field Equations it was felt that they lend an enrichment especially in the discussion on the charged mass.

2. *Uncharged Mass*

In this section we are concerned with a derivation of the Schwarzschild line-element in the vicinity of an uncharged mass. Toward this end we shall introduce the notion of a 'proper system'. Generally, a proper system is a local inertial system which is momentarily at rest with respect to the matter in its immediate vicinity. In our case we may imagine a framework of non-rigid rods (of negligible mass) attached to the central mass. At any point (in space-time) then, a proper system is a local inertial system momentarily at rest with respect to the rod at its location at the moment under consideration. In the following discussion it is important to bear in mind that we are constructing a coordinate system relative to which the proper systems are at rest.†

Consider then a spherically symmetric uncharged mass M, where it is assumed that space is pseudo-Euclidean at great distances from M.‡ It will be useful in the following development to introduce the notion of a 'drift-system'. Drift-systems are infinitesimal cartesian coordinate systems attached to point mass (the mass being very small) particles which are released from rest at infinity at various times and are allowed to drift in toward M. In fact, we assume that there is a spherical layer of such systems at rest at some very great distance from M with M as center. Later we shall consider releasing particular systems at particular times. In the following we shall derive the Schwarzschild line-element by considering the relationship between measurements made in the drift-systems and measurements made in proper systems at the point of interest. Actually, this relationship is used as a vehicle for defining the coordinate system in terms of measurements made solely in the drift-system.

Now consider two near neighboring events (in 4-space) that occur in the vicinity of M. For the time being, we shall only consider events which both have a spatial location (in 3-space) on the 'track' (in 3-space) formed by some drift-system.§ If these events are witnessed in a proper system at the point (in 4-space) of interest we have

$$d\tau^2 = -\Delta_s^{0\,2} + C^2 \Delta_t^{0\,2} \tag{2.1}$$

where $d\tau$ is the world-length between the events and $\Delta_s{}^0$, $\Delta_t{}^0$ denote the spatial distance and time interval respectively between the events as perceived in the proper system. Let Δ_t and Δ_s denote the time and spatial intervals between the same events as perceived in a drift-system which was released from its place of rest at such a time that it arrived at the place of interest at the instant one of the events occurred. It is important to note that, because of the symmetry of the problem, only one of the drift-systems released at some particular time will qualify. Furthermore, the state of motion of a drift-system when it goes through the point of interest will be independent of its time of release, due to the static nature of the situation.

† A clear formulation of this requirement can be found in Møller, C. (1962). *Theory of Relativity*, Chapter 8. Oxford University Press.

‡ The statement that the mass is spherically symmetric will achieve its definition in the course of the following discussion where several assumptions concerning the symmetry of the problem are made.

§ In the following we shall speak of the 3-space projection of the space-time path as the 'drift-track'.

As a guide to our thinking we momentarily restrict considerations to particular pairs of events.

Consider then two neighboring events occurring at the same place and at different times (in the coordinate system under construction) along a drift-track. Then $\Delta_s{}^0 = 0$ if the time interval is sufficiently small. Therefore, we must have

$$\Delta_t = \frac{\Delta_t{}^0}{\sqrt{[1-(V^2/C^2)]}} \tag{2.2}$$

$$d\tau^2 = C^2 \Delta_t^{0\,2} \tag{2.3}$$

where V is the momentary speed of the drift-system relative to the proper system—both systems being momentary Lorentz frames.† We now tentatively *define* the time increment in the coordinate system under construction so that it is given by equation (2.2) in case the events occur at the same place.

Now consider two events that occur (along the same drift-track) at the same time but at different locations (in the coordinate system under construction). In this case we must have

$$\Delta_t{}^0 = 0; \qquad d\tau^2 = -\Delta_s^{0\,2} \tag{2.4}$$

Now, if the length of $\Delta_s{}^0$ measured in the drift-system is called $\tilde{\Delta}_s$, we have

$$\tilde{\Delta}_s = \sqrt{[1-(V^2/C^2)]}\,\Delta_s{}^0 \tag{2.5}$$

where V is the unique speed of the drift-system. Note that $\tilde{\Delta}_s \neq \Delta_s$. In the case of $\tilde{\Delta}_s$, signals are send out from the proper system so that they arrive in the drift-system simultaneously.

If we decided to send out the signals simultaneously from the proper system we would have

$$\tilde{\Delta}_s{}' = \frac{1}{\sqrt{[-(V^2/C^2)]}}\,\Delta_s{}^0 \tag{2.6}$$

Now we shall decide between equations (2.5) and (2.6) as a tentative definition of spatial increment in the case when the events occur simultaneously (in the coordinate system under construction) and at different places. We notice that $\tilde{\Delta}_s{}'$ is not fully determined by measurements in the drift-system alone, since an observer in the proper system must send out signals simultaneously. To determine $\tilde{\Delta}_s$ however, an observer in the drift-system must simply measure the length of a segment of drift-track. This also requires that light signals be sent out from the proper system. But in this case the observer in the proper system need not have any knowledge about the timing of the signals he sends out. We decide, therefore, in favor of equation (2.5) as the definition of coordinate increment for two events occurring at

† Here we are introducing the principle of strong equivalence. Note also, that in view of the preceding discussion V is unique.

different places at the same time, since measurements made solely in the drift-system are involved.†

We now extend our definitions by taking equations (2.3) and (2.5) as the definitions of the increments in time and space for *any* pair of events (along the same drift-track). That is, we are dropping the qualifications that initially accompanied these definitions. In addition, in the case that two events occur at neighboring spatial locations and *arbitrary* times we define $\tilde{\Delta}_s$ in the following way: the locations of the events are held fixed but their times of occurrence are changed until both events 'occur' in the domain of a proper system at the point of interest. $\Delta_s{}^0$ is then recorded and $\tilde{\Delta}_s$ is determined according to equation (2.5).

From equations (1), (2) and (5) we can now write for the world-length between two events (on the same drift-track),

$$d\tau^2 = -\frac{dr^2}{1-(V^2/C^2)} + C^2[1-(V^2/C^2)]\,dt^2 \qquad (2.7)$$

where we have replaced $\tilde{\Delta}_s$ and Δ_t by the more suggestive symbols dr and dt, respectively.

So far we have essentially just defined new coordinate increments which are compatible with the requirement that the proper system at any point be momentarily at rest with respect to the coordinate system under construction. In addition there is a rather subtle assumption involved when we write

$$dr = \tilde{\Delta}_s; \qquad dt = \Delta_t \qquad (2.8)$$

since we are here assuming that $\tilde{\Delta}_s$ and Δ_t are increments in the variables r and t.‡ We will return to this point shortly. For the present we make no use of equation (2.8).

Having arrived at equation (2.7) it seems as if we can go no further without appealing to additional assumptions. We therefore make the following heuristic considerations: when a drift-system is very far from M, we have from Newtonian theory that

$$\tfrac{1}{2}V^2 = \frac{GM}{r} \quad \text{(conservation of energy)} \qquad (2.9)$$

where V and r are the momentary speed and separation of the drift-system relative to M, It is now assumed, under the principle of strong equivalence, that this relation holds exactly for all time as the drift-system approaches M.

In order to utilize this assumption we must redefine V and r so that they have meaning in the context of general relativity. Accordingly, V is defined as the momentary speed of the drift-system relative to the proper system at the point of interest, and r is to be defined by the relation

$$r - \alpha = \sum \tilde{\Delta}_s \qquad (2.10)$$

† In relation to equations (2.6) and (2.2) no concern has been expressed in the literature as to why the radicals should be in the numerator or denominator.

‡ This point does not seem to have been stressed in the literature cited.

where α is a constant that will be determined shortly, and the summation extends from the point in question in toward M (how far in will be determined presently). Then r is, to within an additive constant, the distance as measured by a drift-system observer-traveled from the point in question along the drift-track toward M.

Equation (2.9), together with the included definitions of V and r, comprise our heuristic assumption.

With this assumption we now have

$$1-(V^2/C^2)=1-\frac{2GM}{rC^2} \tag{2.11}$$

so that equation (2.7) becomes

$$d\tau^2=\frac{-dr^2}{1-\dfrac{2GM}{rC^2}}+C^2\left(1-\frac{2GM}{rC^2}\right)dt^2 \tag{2.12}$$

which is the Schwarzschild line-element for events on the same drift-track.

There are several points needing clarification. First, we can easily evaluate the constant α appearing in equation (2.10) by combining equations (2.5), (2.9) and (2.10) to give

$$r-\alpha=\sum\sqrt{[1-(V^2/C^2)]}\,\Delta_s{}^0=\sum\sqrt{\left(1-\frac{2GM}{rC^2}\right)}\,\Delta_s{}^0 \tag{2.13}$$

where the summation extends from the point in question in toward M—as far as possible—which is seen to be the point $r=r_0$, where

$$r_0=\frac{2GM}{C^2} \tag{2.14}$$

This is also the point where $V=C$.

It follows then that

$$\alpha=r_0=\frac{2GM}{C^2} \tag{2.15}$$

and r is fully defined. We notice of course, that the gravitational singularity (at $r=\alpha=r_0$) very naturally enters the construction, and further, it is just the point where V becomes equal to C.†

Secondly, we must consider the problems of the *differentials dr* and *dt*.

Concerning dr: r was actually defined by the relation (equation (2.10)]

$$r-\alpha=\sum\tilde{\Delta}_s$$

In order to be able to consider $dr=\tilde{\Delta}_s$ it must be that for two events occurring at neighboring locations (along a drift-track) and *arbitrary* times the difference in their r values is just the corresponding $\tilde{\Delta}_s$. According to the discussion

† In contrast, McVittie defines a non-physical speed, q, in such a manner that $q=0$ at $r=r_0$. See McVittie, G. C. (1962). *General Relativity and Cosmology*, p. 85. The University of Illinois Press.

prior to equation (2.7) the $\tilde{\Delta}_s$ are completely independent of the times at which the events occur. Therefore, the relation $\tilde{\Delta}_s = dr$ is valid, where r is given by relation (2.10).

Concerning dt: here the problem is somewhat different. We haven't yet defined a coordinate t whose change can be identified with what we have defined as dt. A possible way of defining t might be as follows: we choose a zero time at infinity (where the drift-systems are initially at rest and space is Galilean and all clocks can be synchronized). We then release any particular drift-system at such a time that it goes through the point of interest at the instant the event in question occurs. We might define t by the relation

$$t = \sum \frac{\Delta_t^0}{\sqrt{[1 - (V^2/C^2)]}} \tag{2.16}$$

where the summation extends from zero time to the time the event in question occurs, as the drift-system goes from its initial location to the location of the event. The Δ_t^0 here signify the successive time increments measured in the appropriate proper systems situated along the drift-track of the incoming drift-system.

Unfortunately, if we apply this definition to two neighboring events that occur along a drift-track at such times that a drift-system cannot travel from one location to the other in the allowed time interval, then the differences in these times is *not* given by the above summand. The proof goes as follows:

Consider a given drift-track upon which two neighboring events occur, at locations 1 and 2 at the times t_1 and t_2 respectively. Then we can write

$$t_1 = t_1^{\infty} + t_1^{\text{in}}; \qquad t_2 = t_2^{\infty} + t_2^{\text{in}} \tag{2.17}$$

where $t_1^{\infty}(t_2^{\infty})$ is the time spent at infinity (after zero time) before the drift-system is released which arrives at location 1 (2) at time $t_1(t_2)$; and $t_1^{\text{in}}(t_2^{\text{in}})$ is the time required for the drift-system after release to arrive at the location 1 (2)—for example

$$t_1^{\text{in}} = \sum \frac{\Delta_t^0}{[1 - (V^2/C^2)]} \tag{2.18}$$

the summation beginning at the time the drift-system is released and ending at the time of its arrival at the location in question. Now consider the same drift-track again with the same event occurring at location 1 at time t_1. Let the drift-system which arrives at location 1 continue on until it arrives at the same location 2—which will be at the new time say, t_2'. The events at times t_1 and t_2' are therefore 'connected' by a drift-system. Now we have

$$t_2' = t_2'^{\infty} + t_2'^{\text{in}} \tag{2.19}$$

and since location 2 is the same in both cases we must have

$$t_2'^{\text{in}} = t_2^{\text{in}} \tag{2.20}$$

In the latter case we must have

$$t_2' - t_1 = t_2'^{\text{in}} - t_1^{\text{in}} = \Delta_t \tag{2.21}$$

since these points are 'connected' by a drift-system. Therefore we have

$$(t_2 - t_1) = (t_2^\infty - t_1^\infty) + (t_2'^{\text{in}} - t_1^{\text{in}}) = (t_2^\infty - t_1^\infty) + \Delta_t \tag{2.22}$$

Now when the events at location 1 and 2 are not connected by a drift-system we know that

$$t_1^\infty - t_2^\infty \neq 0 \tag{2.23}$$

Therefore in this case

$$\Delta_t \neq (t_2 - t_1) \tag{2.24}$$

which was the statement to be proven.†

There are other possible ways one might define t but they meet with the same objection. This problem seems related to the fact that in order to associate a time with an event, a drift-system must pass through the point in question at the instant the event occurs. There are thus two requirements to be met. In the definition of r and dr the only requirement was that the drift-system pass through the point of interest—at any time. This difference in turn is apparently due to the fact that we can preserve in this static problem—for all time the location of an event by merely requiring the event to leave a 'spot' at its location of occurrence. However, there seems to be no way of preserving the time of an event over all space, that is, of identifying the same time at different points of space.

In light of this discussion, the term involving dt^2 in equation (2.12) must be interpreted carefully. In general, we do not even attempt to define a t for an event. We only consider Δ_t, which is not necessarily a time increment —but what we may call a time duration. We can still, however, treat Δ_t as the differential of a time coordinate in certain cases—like the motion of a drift-system. In describing such a process it is legitimate to define dt by the relation

$$dt = \Delta_t \tag{2.25}$$

This is permissible here since we are *only* considering pairs of events 'connected' by drift-system motions.

3. *Rigorous Considerations*

In the preceeding considerations the form of the Schwarzschild line-element was constructed. The argument was rigorous except at the point where the heuristic assumption that

$$\tfrac{1}{2}V^2 = \frac{GM}{r}$$

† This point does not appear in the literature cited.

was made, with a particular interpretation given to V and r. It will now be demonstrated that equation (2.9), together with the assumed interpretation of V and r, follow from the Field Equations and the Equations of Motion.

Now, given a spherically symmetric uncharged mass M, and requiring that space be Galilean at great distances, one has, according to Schwarzschild (Jeffrey, 1921), the following expression for the line-element

$$d\tau^2 = \frac{-dr^2}{1-\dfrac{2GM}{rC^2}} - r^2\sin^2\theta\, d\phi^2 - r^2\, d\theta^2 + C^2\left(1-\frac{2GM}{rC^2}\right)dt^2 \tag{3.1}$$

At present no physical interpretation for r and t is assumed,† the above expression being just a formal solution to the Field Equations.

Using the metric from this expression in the geodesic equations of motion one obtains the following relations‡

$$\frac{d^2 r}{d\tau^2} + \frac{1}{2}\frac{d\lambda}{dr}\left(\frac{dr}{d\tau}\right)^2 - r\exp(-\lambda)\left(\frac{d\theta}{d\tau}\right)^2 - r\sin^2\theta\exp(-\lambda)\times$$
$$\times\left(\frac{d\phi}{d\tau}\right)^2 + \frac{C^2\exp(\nu-\lambda)}{2}\frac{d\nu}{dr}\left(\frac{dt}{d\tau}\right)^2 = 0$$

$$\frac{d^2\theta}{d\tau^2} + \frac{2}{r}\frac{dr}{d\tau}\frac{d\theta}{d\tau} - \sin\theta\cos\theta\left(\frac{d\phi}{d\tau}\right)^2 = 0$$

$$\frac{d^2\phi}{d\tau^2} + \frac{2}{r}\frac{dr}{d\tau}\frac{d\phi}{d\tau} + 2\cot\theta\frac{d\phi}{d\tau}\frac{d\theta}{d\tau} = 0$$

$$\frac{d^2 t}{d\tau^2} + \frac{d\nu}{dr}\frac{dr}{d\tau}\frac{dt}{d\tau} = 0 \tag{3.2}$$

where

$$e^{\nu} = \exp(-\lambda) = \left(1-\frac{2GM}{rC^2}\right) \tag{3.3}$$

Then for a freely moving point particle which is released from rest at any finite r we have $d\phi/d\tau = dr/d\tau = d\theta/d\tau = 0$ at the point of release. From the second and fourth equations above, this implies that for such a system $d\phi/d\tau = 0$ and $d\theta/d\tau = 0$ throughout the entire motion. Therefore, drift-systems are characterized by constant θ and ϕ. One then obtains as the first

† It is being assumed, however, that the Schwarzschild coordinates (r, θ, ϕ) of any point which is fixed relative to our framework of rods connected to M are constant in Schwarzschild time. This implies that the proper systems introduced earlier are the same as the local inertial systems momentarily at rest relative to the above formal Schwarzschild coordinate system.

‡ Tolman, R. C. (1958). *Relativity Thermodynamics and Cosmology*, pp. 206–207. Oxford University Press.

integrals of the motion from the first and fourth equations above, the relations

$$e^{\lambda}\left(\frac{dr}{d\tau}\right)^2 - e^{\nu} C^2 \left(\frac{dt}{d\tau}\right)^2 + 1 = 0$$

$$C\left(\frac{dt}{d\tau}\right) = k \exp(-\nu) \tag{3.4}$$

where k is a constant.

Now, letting $r \to \infty$, corresponding to our definition of a drift-system, we have $\exp(-\nu) \to 1$ and $dr/dt \to 0$. Further then, as $r \to \infty$, $dt/d\tau \to C^{-1}$. Therefore, the second of equations (3.4) gives

$$k = 1 \tag{3.5}$$

for drift-systems. Therefore, the first of equations (3.4) becomes

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{2GM}{rC^2} \tag{3.6}$$

Now let Δ_s^0, Δ_t^0 denote distances and times associated with the incoming drift-system as measured in the proper system. From equation (3.1) as applied to a drift-system we have

$$\Delta_s^0 = \frac{dr}{\sqrt{\left(1 - \dfrac{2GM}{rC^2}\right)}}, \qquad \Delta_t^0 = \frac{dt}{\sqrt{\left(1 - \dfrac{2GM}{rC^2}\right)}} \tag{3.7}$$

since $d\theta = 0$, $d\phi = 0$ for a drift-system. The speed of the drift-system relative to the proper-system at any particular point is given by

$$V \equiv \frac{\Delta_s^0}{\Delta_t^0} = \left(1 - \frac{2GM}{rC^2}\right)^{-1} \left(\frac{dr}{dt}\right) \tag{3.8}$$

Now,

$$\frac{dr}{d\tau} = \frac{dr}{dt}\frac{dt}{d\tau}; \qquad \frac{dt}{d\tau} = \frac{1}{C}\exp(-\nu) \tag{3.9}$$

from the second of equations (3.4). Therefore,

$$\left(\frac{dr}{dt}\right)^2 = C^2 \exp(2\nu) \left(\frac{dr}{d\tau}\right)^2 \tag{3.10}$$

Now combining equations (3.8), and (3.9) and (3.10) we obtain

$$V^2 = \frac{2GM}{r} \tag{3.11}$$

or

$$\tfrac{1}{2}V^2 = \frac{GM}{r} \tag{3.12}$$

which was the equation (2.9) to be derived.†

Further, referring back to equations (3.7) we can now write

$$dr = \sqrt{[1-(V^2/C^2)]}\,\Delta_s^0, \qquad dt = \frac{\Delta_t^0}{\sqrt{[1-(V^2/C^2)]}} \tag{3.13}$$

Therefore,

$$r + \text{constant} = \sum \tilde{\Delta}_s \tag{3.14}$$

in agreement with the defining equation (2.10) on r.

In summary, we have constructed the Schwarzschild line-element about a symmetric uncharged mass as it pertains to any pair of events on the same drift-track. The coordinate r and its differential dr have been given full significance in terms of measurements performed with physical rods by an observer in the drift-system. In these measurements the proper system does not have to be consulted.‡ In this sense we have not found the full significance of t or dt. First, because we have not found a way of determining t in terms of rod and clock measurements depending solely on measurements performed in the drift-system; and second, because in general we cannot identify a change in t with the quantity dt as measured in a drift-system.

Therefore, we have only succeeded in giving full significance to the r coordinate.

The restriction to events occurring on the same drift-track will be removed after considering the case of the charged mass. The customary full form of the line-element will then have been constructed for the cases of the charged and uncharged mass—with full significance having been found for the spatial part of the line-element.

4. *Charged Mass*

In this section we are interested in constructing—with physical significance—the Schwarzschild line-element about a spherical mass M carrying a uniform charge q. Since these considerations will be so similar to those for the uncharged case, the discussion will be quite brief.

In analogy to the preceding section, we first make the following heuristic assumptions:

(1) The total gravitating power of the charged mass is determined by M. That is, the gravitating power of the charge q is already included in what we call M.

† Using these same relations it is easy to show for a drift-system released at any speed that $v = C$ at $r = r_0$.

‡ That is, the drift-system observer needs no measuremental results from the proper system in order to make his own measurement.

(2) Conservation of energy, applied to a drift-system, holds in the classical form with suitable interpretations given to the variables involved.

We shall first apply (1) and (2) to see what form conservation of energy takes for a drift-system. For the moment, the argument is essentially Newtonian. Consider a drift-system approaching M. At some instant when their mutual separation is r, the drift-system will be exposed to the gravitational action of M *minus* the gravitational mass of the electric field already penetrated. This follows from (1). The mass of the electric field already penetrated is easily calculated as follows: the electrostatic field energy W beyond r is given by

$$W = \frac{1}{8\pi} \int E^2 \, d^3 x = \frac{q^2}{2r} \tag{4.1}$$

The gravitational mass associated with this energy is given by

$$\frac{q^2}{2rC^2} . 2 = \frac{q^2}{rC^2} \tag{4.2}$$

The factor two appears since we know from other considerations† that electromagnetic fields are twice as effective gravitatively as mechanical mass.

Therefore, when the separation between the drift-system and M is r, the force of attraction on the drift-system (divided by its mass) is

$$G\left(M - \frac{q^2}{rC^2}\right)\frac{1}{r^2} \tag{4.3}$$

Equating the work done (per unit mass) to the change in kinetic energy (per unit mass) of the drift-system we have

$$\int_r^{\infty} G\left(M - \frac{q^2}{rC^2}\right)\frac{1}{r^2} dr = \tfrac{1}{2}V^2 \tag{4.4}$$

And this gives the relation

$$1 - (V^2/C^2) = 1 - \frac{2GM}{rC^2} + \frac{Gq^2}{r^2 C^4} \tag{4.5}$$

Now, proceeding exactly as in the case of the uncharged mass, we obtain

$$d\tau^2 = -\frac{dr^2}{1 - \dfrac{2GM}{rC^2} + \dfrac{Gq^2}{r^2 C^4}} + C^2\left(1 - \frac{2GM}{rC^2} + \frac{Gq^2}{r^2 C^4}\right) dt^2 \tag{4.6}$$

which is the line-element about a charged mass as derived by Jeffrey (1921).

Finally, it is also straightforward to demonstrate that assumptions (1)

† See for instance, Tolman, R. C. (1958). *Relativity Thermodynamics and Cosmology*, p. 285. Oxford University Press.

and (2) are valid consequences of the Field Equations, and the geodesic equations of motion as applied to the (neutral) drift-systems.

5. *Extension*

The preceding discussion consisted of a construction of the line-element about an uncharged and charged mass as it pertains to any pair of neighboring events whose spatial locations are on the same drift-track. We now finally note that since the drift-systems are released from a sphere of such size that it lies in the region of space which is Galilean, we may immediately generalize the derived line-elements so that they apply to *any* pair of neighboring events by just adding the term, $-r^2 d\theta^2 - r^2 \sin^2\theta\, d\phi^2$, to $d\tau^2$, where θ and ϕ have their customary significance.

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